

# The cross section of a spherical double cone

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## Abstract

We investigate the poset of  $SL(n)$  orbit closures in the product of two partial flag varieties. In particular we prove that it is always a lattice.

**Keywords:** Spherical double cones, partial flag varieties, ladder posets.

**MSC:** 06A07, 14M15

## 1 Introduction

Let  $G$  be a reductive algebraic group and let  $X$  be a  $G$  variety.  $X$  is called a spherical  $G$  variety, or  $G$ -spherical, if there exists a Borel subgroup of  $G$  with a dense open orbit in  $X$ .

Let  $P_1, \dots, P_k \subset G$  be a list of parabolic subgroups containing the same Borel subgroup  $B$ . The product

$$X = G/P_1 \times \cdots \times G/P_k \tag{1}$$

is a  $G$  variety via diagonal action. Determining when  $X$  is a spherical  $G$  variety carries important information for representation theory, in particular for understanding multiplicity free representations of  $G$ . In his ground breaking article [5], Littelmann initiated the classification problem and gave a list of all possible pairs of maximal parabolic subgroups  $P_1, P_2$  such that  $G/P_1 \times G/P_2$  is  $G$ -spherical. In [6], for group  $G = SL(n)$  and in [7] for  $G = Sp(2n)$ , Magyar, Weyman, and Zelevinski classified the parabolic subgroups  $P_1, \dots, P_k$  such that the product  $X = G/P_1 \times \cdots \times G/P_k$  is  $G$ -spherical. It turns out, according to [6], for spherical  $X$ , the number of factors is at most 3, and  $k = 3$  occurs in only special cases. Therefore, the gist of the problem lies in the case  $k = 2$ . This case is settled in full detail by Stembridge in [10]. More precisely, in [10] Stembridge gave a complete list of all parabolic subgroups  $P_i \subset G$ ,  $i = 1, 2$ , where  $G$  is a semisimple Lie group of any type such that  $G/P_1 \times G/P_2$  is spherical.

Let us call a partial flag variety of the form  $G/P$ , a  $G$ -flag variety. Of course, there is no particular reason for confining ourselves to the products of  $G$ -flag varieties only. Indeed, there are very important variations of this theme. Let  $K \subset G$  be a symmetric subgroup of a connected simple algebraic group. (A subgroup  $K \subset G$  is called symmetric if there exists an involutory automorphism  $\theta : G \rightarrow G$  such that  $K = \{g \in G : \theta(g) = g\}$ .) Let  $B_K$  be a Borel subgroup of  $K$  and let  $P \subset G$  be a parabolic subgroup. The natural related question is

*Question:* What are the conditions on  $(G, K, P)$  so that the “ $G \times K$ -flag variety”  $G/P \times K/B_K$  is  $G$ -spherical?

It turns out that a classification of such triplets  $(G, K, P)$  is equivalent to the classification of  $K$ -spherical  $G$ -flag varieties. A proof of the equivalence, as well as classification of these triplets is given in [3]. More recently, towards the goal of better understanding vector valued orthogonal polynomials, van Pruijssen [12] has extended the classification in [3] to the case when  $H$  is an arbitrary connected reductive subgroup. See also [8]. Let us also mention that, we recently managed to classify the triplets  $(G, K, P)$  where  $K$  is a symmetric subgroup,  $P$  is a parabolic subgroup and  $G/K \times G/P$  is a spherical  $G$ -variety, [1].

From representation theoretic point of view, the first mentioned classification problem amounts to understanding of the ring of invariants of a maximal unipotent subgroup in the coordinate ring of  $X = G/P_1 \times G/P_2$ . This problem, in turn, is closely related to the combinatorics of the  $G$  orbits in  $X$  (see [5]). Our goal in this note is to prove the following result on  $G$ -orbits.

**Theorem 1.1.** Let  $G$  denote the special linear group  $SL(n)$  and let  $P_1$  and  $P_2$  be two parabolic subgroups. If the  $G$  action on  $G/P_1 \times G/P_2$  is spherical, then the inclusion poset on  $G$  orbit closures is a lattice.

More precise description of these lattices is given in Section 4.

## 2 Preliminaries

For simplicity we assume that  $G$  is simple and simply connected. Let  $B = UT \subset G$  be a Borel subgroup with the maximal unipotent subgroup  $U \subset B$  and a maximal torus  $T \subset B$ .

### 2.1

Let  $\Phi$  be the root system determined by  $(G, T)$  let  $\Delta = \{\alpha_1, \dots, \alpha_n\} \subset \Phi$  denote its system of simple roots relative to  $B$  and let  $\omega_1, \dots, \omega_n$  denote the associated fundamental dominant weights, which form a basis for the weight lattice. In particular, for each  $j = 1, \dots, n$ , there exists an irreducible representation of  $G$ , denoted by  $V(\omega_j)$ , and a line  $\ell \subset V(\omega_j)$  such that the stabilizer of  $\ell \in \mathbb{P}(V(\omega_j))$  is a maximal parabolic subgroup  $P_j \subset G$ . In particular we have the canonical embedding  $G/P_j \hookrightarrow \mathbb{P}(V(\omega_j))$ .

Let  $\mathcal{C}_j \subset V(\omega_j)$  denote the cone over  $G/P_j$  and set  $\mathcal{C}_{i,j} := \mathcal{C}_i \times \mathcal{C}_j$ . In [5], Littelmann shows that for diagonal action of  $U$  the  $U$ -invariant functions on  $\mathbb{C}[\mathcal{C}_{i,j}]$  is a polynomial ring and he classifies all pairs  $(\omega_i, \omega_j)$  such that  $\mathbb{C}[\mathcal{C}_{i,j}]^U$  is freely generated by elements of linearly independent weights. This amounts to classifying spherical products  $G/P_i \times G/P_j$  of partial flag varieties attached to fundamental weights.

## 2.2

Suppose  $G$  acts on two irreducible varieties  $X_1$  and  $X_2$ , and  $x_i \in X_i$ ,  $i = 1, 2$  be points in general position. If  $G_i \subset G$  denotes the stabilizer subgroup of  $x_i$  in  $G$ , then  $\text{Stab}_G(x_1 \times x_2)$  coincides with the stabilizer in  $G_1$  of a point in general position from  $G/G_2$  (or, equivalently, with the stabilizer in  $G_2$  of a point in general position from  $G/G_1$ ). Consequently, studying  $G$  orbits in a spherical product  $G/P_1 \times G/P_2$  reduces to the study of  $P_1$  orbits in  $G/P_2$ .

**Remark 2.1.** 1. The parabolic group  $P_1$  acts on the product  $G \times G/P_2$  by  $g \cdot (g_1, xP_2) = (g_1g^{-1}, gxP_2)$ . This action is free; we denote the quotient by  $G \times^{P_1} G/P_2$ . Thus, two points  $x, y \in G \times G/P_2$  projects to represent the same point  $[x] = [y]$  in  $G \times^{P_1} G/P_2$  if they are of the form  $x = (g_1, x_1P_2)$ ,  $y = (g_2, x_2P_2)$  and there exists  $g \in P_1$  such that  $g_1 = g_2g^{-1}$  and  $x_1P_2 = gx_2P_2$ .

2. Extending the action of  $P_1$  on  $G \times G/P_2$ , the whole group  $G$  acts on  $G \times^{P_1} G/P_2$ . Then the map

$$G \times^{P_1} G/P_2 \rightarrow X = P_1 \backslash G \times G/P_2, \quad [(g, qP_2)] \mapsto (P_1g^{-1}, gqP_2) \quad (2)$$

is a  $G$  equivariant isomorphism.

So our objective is to understand  $P_1$  orbits in  $G/P_2$ . It is well known that these orbits are parametrized by  $(W_1, W_2)$ -double cosets in  $W$ , where  $W_1, W_2$ , and  $W$  are the Weyl groups of  $P_1, P_2$  and  $G$ , respectively. The Bruhat order on  $W_1 \backslash W / W_2$  is well-known. In fact, it has various characterizations (see [11, Proposition 1.8]). Let

$$\pi : W \rightarrow W_1 \backslash W / W_2$$

denote the canonical projection onto double cosets. It turns out that the preimage in  $W$  of every double coset in  $W_1 \backslash W / W_2$  is an interval with respect to Bruhat order, hence it has a unique maximal and a unique minimal element. Moreover, if  $[w_1], [w_2] \in W_1 \backslash W / W_2$  are two double cosets represented by the maximal elements  $w_1, w_2$  of the corresponding intervals, then  $[w_1] \leq [w_2]$  in  $W_1 \backslash W / W_2$  if and only if  $w_1 \leq w_2$  in  $W$ . (See [4].) Although it looks as if this observation settles the question of inclusion order on closures of  $G$  orbits, understanding the exact nature of this poset is very much desirable for better understanding the products.

## 2.3

Without loss of generality we assume that  $T$  is contained in both of the parabolic subgroups  $P_1$  and  $P_2$ . Also, let  $B$  be a Borel subgroup such that  $T \subset B \subset P_1 \cap P_2$ . Hence,  $P_2$

and  $P_1$  are uniquely determined by subsets  $J$  and  $I$ , respectively, of the set of simple roots  $\Delta = \Delta(G, B, T)$  of  $\Phi$ . In particular, the Weyl (sub)groups corresponding to  $P_2$  and  $P_1$ , denoted by  $W_J$  and  $W_I$ , respectively, are generated by the simple reflections in the sets  $I, J$ , respectively. We reserve the letter  $R$  for the Coxeter generators  $R = R(\Delta)$  of  $W$ . (Thus,  $I$  and  $J$  are subsets of  $R$ , rather than subsets of  $\Delta$ ; however, whenever it is convenient we identify  $I$  (and  $J$ ) with the corresponding set  $\tilde{I} \subset \Delta$  of simple roots. We prefer to use this notation since the action of  $W$  on these sets is more clear in the context of Weyl groups.) Also, from now on we are going to write  $P_I$  and  $P_J$  instead of  $P_1$  and  $P_2$ , respectively.

The set of distinguished  $(W_I, W_J)$ -double coset representatives is defined as follows. Let  $[w]$  be a representative of a double coset in  $W_I \backslash W / W_J$  such that  $\ell(w)$  is as small as possible. It turns out that such  $w \in W$  are characterized by  $w \in {}^I W \cap W^J$ , where  ${}^I W$  is the set of minimal length coset representatives for  $W_I \backslash W$ . We denote  ${}^I W \cap W^J$  by  $X_{I,J}^-$ . Set  $H = I \cap wJw^{-1}$ . Then  $uw \in W^J$  for  $u \in W_I$  if and only if  $u$  is a minimal length coset representative for  $W_I / W_H$ . In particular, every element of  $W_I w W_J$  has a unique expression of the form  $uwv$  with  $u \in W_I$  is a minimal length coset representative of  $W_I / W_H$ ,  $v \in W_J$  and  $\ell(uwv) = \ell(u) + \ell(w) + \ell(v)$ .

Another characterization of the sets  $X_{I,J}^-$  is as follows. For  $w \in W$ , the *right ascent set* is defined as

$$\text{Asc}_R(w) = \{s \in R : \ell(ws) > \ell(w)\}.$$

The *right descent set*,  $\text{Des}_R(w)$  is the complement  $R - \text{Asc}_R(w)$ . Similarly, the *left ascent set* of  $w$  is

$$\text{Asc}_L(w) = \{s \in R : \ell(sw) > \ell(w)\} \quad (= \text{Asc}_R(w^{-1})).$$

Then

$$X_{I,J}^- = \{w \in W : I \subseteq \text{Asc}_L(w) \text{ and } J \subseteq \text{Asc}_R(w)\} \quad (3)$$

$$= \{w \in W : I^c \supseteq \text{Des}_R(w^{-1}) \text{ and } J^c \supseteq \text{Des}_R(w)\} \quad (4)$$

For our purposes we need the distinguished set of maximal length representatives for each double coset.

$$X_{I,J}^+ = \{w \in W : I \subseteq \text{Des}_R(w^{-1}) \text{ and } J \subseteq \text{Des}_R(w)\} \quad (5)$$

$$= \{w \in W : I^c \supseteq \text{Asc}_R(w^{-1}) \text{ and } J^c \supseteq \text{Asc}_R(w)\} \quad (6)$$

For a proof of this characterization of  $X_{I,J}^+$  see [2, Theorem 1.2(i)].

**Remark 2.2.** The Bruhat orders on  $X_{I,J}^-$  and  $X_{I,J}^+$  are isomorphic. Indeed, the spherical products  $P_I \backslash G \times G / P_J$  and  $P_I^{op} \backslash G \times G / P_J^{op}$  are isomorphic, where  $P_I^{op}$  stands for the opposite parabolic to  $P_I$ .

### 3 Tight Bruhat posets

We already know the classification of parabolic subgroups  $P_I, P_J$  corresponding to fundamental dominant weights such that the diagonal  $G$  action on  $G/P_I \times G/P_J$  is spherical. Said differently, we know all pairs  $(I, J)$  of subsets of Coxeter generators  $R \subset W$  such that

- $|I| = |J| = |R| - 1$ , and
- $G/P_I \times G/P_J$  is  $G$  spherical.

In particular, under the maximality assumption on subsets  $I$  and  $J$ , the poset of  $G$ -orbit closures is always a chain, see [5, Proposition 3.2]. In the light of our remarks above, this is equivalent to showing that  $W_I \backslash W / W_J$  is a chain.

As we mentioned earlier, the classification of Littelmann is extended by Stembridge to cover all pairs of subsets  $(I, J)$  in  $R$  such that  $G/P_I \times G/P_J$  is  $G$  spherical. See Corollaries 1.3.A – 1.3.D, 1.3.E6, 1.3.E7, and 1.3.{E8,F4,G2} in [10].

**Remark 3.1.** 1. In the cases of A–D, E6, and E7, if  $G/P_I \times G/P_J \neq G/B \times G/B$  is a spherical product, then at least one of  $I$  and  $J$  is maximal, that is to say, of cardinality  $|R| - 1$ . Without loss of generality we always choose  $I$  to be the maximal one.

Next, we review the useful concept of *tight Bruhat order*. Let  $\mathbf{E}$  be a real vector space of smallest dimension containing our root system  $\Phi$ . Let  $\Delta \subset \Phi$  be a basis for  $\mathbf{E}$  (hence,  $\Delta$  is a set of simple roots),  $(W, R)$  be the associated Coxeter system. We are interested in the case when  $W$  is a Weyl group. One way to define the Bruhat-Chevalley order on  $W$  is to use the reflection representation of  $W$  as the group of isometries of  $\mathbf{E}$ : let  $\langle \cdot, \cdot \rangle$  denote the  $W$ -invariant inner product on  $\mathbf{E}$ , and let  $\theta \in \mathbf{E}$  be a vector such that  $\langle \theta, \beta \rangle \geq 0$  for all  $\beta \in \Phi^+$ . Such a vector is called dominant. It is well-known that the stabilizer of a dominant vector is a parabolic subgroup  $W_J \subset W$ , where  $J = \{s_\alpha \in R : \langle \theta, \alpha \rangle = 0\}$ . Thus, as a set, the minimal length coset representatives  $W^J \subset W$  of the quotient  $W/W_J$  can be identified with the orbit  $W \cdot \theta$ . Following Stembridge, we are going to call the orbit map  $w \mapsto w \cdot \theta$  the evaluation.

A proof of the following result can be found in [9]:

**Proposition 3.2.** Let  $\theta \in \mathbf{E}$  be a dominant vector with stabilizer  $W_J$ . The evaluation map induces a poset isomorphism between the Bruhat-Chevalley order on  $W^J$  and the orbit  $W\theta$  with partial order defined by the transitive closure of the relations

$$\mu <_B s_\beta(\mu) \text{ for all } \beta \in \Phi^+ \text{ such that } \langle \mu, \beta \rangle > 0.$$

Now, let  $I \subset R$  be a subset of the Coxeter generators for  $W$ , and let  $\Phi_I \subset \Phi$  denote the root subsystem corresponding to parabolic subgroup  $W_I$ . Accordingly, let  $\Phi_I^+$  denote  $\Phi^+ \cap \Phi_I$ . If  $\theta$  is a dominant vector and its stabilizer subgroup is  $W_J$  with  $J \subset R$ , then define

$$(W\theta)_I := \{\mu \in W\theta : \langle \mu, \beta \rangle \geq 0 \text{ for all } \beta \in \Phi_I^+\}. \quad (7)$$

A proof of the following result can be found in [11, Proposition 1.5].

**Proposition 3.3.** Let  $I, J \subset R$  be two sets of Coxeter generators for  $W$  and let  $\theta \in \mathbf{E}$  be a dominant vector with stabilizer  $W_J$ . Then the evaluation map induces a poset isomorphism between the (restriction of) Bruhat-Chevalley order on  $X_{I,J}^-$  and  $(W\theta)_I$  with partial order defined by the transitive closure of the relations

$$\mu <_B s_\beta(\mu) \text{ for all } \beta \in \Phi^+ \text{ such that } s_\beta(\mu) \in (W\theta)_I \text{ and } \langle \mu, \beta \rangle > 0.$$

Now we come to the definition of a critical notion for our proof. There is a natural partial ordering on the roots defined by

$$\nu \preceq \mu \iff \mu - \nu \in \mathbb{R}^+ \Phi^+. \quad (8)$$

It turns out, when the interpretation of Bruhat-Chevalley ordering as given in Proposition 3.2 is used, there is a natural order reversing implication:

$$\mu \leq_B \nu \implies \nu \preceq \mu. \quad (9)$$

If the converse implication also holds, then the poset  $W\theta$  is called tight. More precisely, a subposet  $(M, \leq_B)$  of the Bruhat-Chevalley order on  $(W\theta, \leq_B)$  is called tight if

$$\mu \leq_B \nu \iff \nu \preceq \mu$$

for all  $\nu, \mu$  in  $M \subset \mathbf{E}$ .

In the light of our Remark 3.1, we assume that  $I \subset R$  is a maximal subset, that it is of the form  $I = R - \{s\}$  for some  $s \in R$ . Also, we assume that there exists a dominant  $\theta \in \mathbf{E}$  such that  $W_J$  is its stabilizer subgroup.

Now, by [11, Theorem 2.3], we see that if  $W^J$  is tight, then  $X_{I,J}^- = X_{R-\{s\},J}^-$  is a chain. The list of tight quotients is also given in [11];  $(W^J, \leq_B)$  is tight if and only if  $W$  is of at most rank 2, or  $J = R$ , or one of the following holds:

- $W \cong A_n$  and  $J^c = \{s_j\}$  ( $1 \leq j \leq n$ ) or  $J^c = \{s_j, s_{j+1}\}$  ( $1 \leq j \leq n-1$ ),
- $W \cong B_n$  and  $J^c = \{s_1\}, \{s_2\}, \{s_n\}$ , or  $J^c = \{s_1, s_2\}$ ,
- $W \cong D_n$  and  $J^c = \{s_1\}, \{s_2\}$  or  $J^c = \{s_n\}$ ,
- $W \cong E_6$  and  $J^c = \{s_1\}$  or  $J^c = \{s_6\}$ ,
- $W \cong E_7$  and  $J^c = \{s_7\}$ ,
- $W \cong F_4$  and  $J^c = \{s_1\}$  or  $J^c = \{s_4\}$ , or
- $W \cong H_3$  and  $J^c = \{s_1\}$  or  $J^c = \{s_3\}$ .

Therefore, in these cases (when  $I$  is maximal and  $J$  is as in this list), then we know that  $X_{I,J}^- = X_{R-\{s\},J}^-$  is a chain. Here is the list of the remaining cases under the assumption that  $I$  is of the form  $R - \{s\}$  for some  $s \in R$ :

- $W \cong A_n$ 
  1.  $I^c = \{s_2\}$  or  $\{s_{n-1}\}$  and  $J^c = \{s_p, s_q\}$  ( $1 < p < p+1 < q < n$ ),
  2.  $|I^c| = 1$  and  $J^c = \{s_1, s_j\}$  or  $\{s_j, s_n\}$  with  $2 < j < n-1$ .
- $W \cong B_n$ 
  1.  $I^c = \{s_n\}$  and  $|J^c| = 1$ .
- $W \cong C_n$ 
  1.  $I^c = \{s_n\}$  and  $|J^c| = 1$ .
- $W \cong D_n$  ( $n \geq 4$ )
  1.  $I^c = \{s_n\}$  and  $J^c = \{s_i\}$  ( $1 < i < n$ ) or  $J^c = \{s_l, s_i\}$  ( $1 \leq i \leq n, l = 1, 2$ ),
  2.  $I^c = \{s_1\}$  or  $\{s_2\}$  and  $J^c \subsetneq \{s_1, s_2, s_n\}$  or  $J^c \subset \{s_{n-1}, s_n\}$  or  $J^c = \{s_{n-2}\}$ .  
(Exclude the case  $J^c = \{n\}$ .)
  3. ( $n = 4$  case only)  $I^c = \{s_1\}$  and  $J^c = \{s_2, s_3\}$  or  $I^c = \{s_2\}$  and  $J^c = \{s_1, s_3\}$ .
- $W \cong E_6$ 
  1.  $I^c = \{s_1\}$  or  $\{s_6\}$  and  $J^c = \{s_i\}$  ( $i = 2, 3, 5$ ) or  $J^c = \{s_1, s_6\}$ .

We are going to analyze the type  $A$  case in the next section.

## 4 Main result

*Notation change: For simplicity of our notation from now on we identify a simple reflection  $s_i$  with its index  $i$ .*

### 4.1 Case 1.

We start with the case  $I^c = \{2\}$ .

Let  $w = w_1 \dots w_{n+1}$  be an element, in one-line notation, from  $X_{I,J}^+$ . Recall that

$$X_{I,J}^+ = \{w \in W : I^c \supseteq \text{Asc}_R(w^{-1}) \text{ and } J^c \supseteq \text{Asc}_R(w)\}$$

The meaning of  $I^c = \{2\} \supseteq \text{Asc}_R(w^{-1})$  is that either  $\text{Asc}_R(w^{-1}) = \emptyset$ , in which case  $w$  is equal to  $w_0$ , the longest permutation, or,  $\text{Asc}_R(w^{-1}) = \{2\}$  hence 2 comes before 3 in  $w$ . Similarly,  $\text{Asc}_R(w)$  cannot be empty unless  $X_{I,J}^+ = \{w_0\}$ .

We continue with the assumption that  $w \neq w_0$ . Suppose  $J^c = \{p, q\}$  for  $1 < p < p+1 < q < n$ . We are going to write  $L_1$  for the segment  $w_1 w_2 \dots w_{p-1}$ ,  $L_2$  for the segment  $w_p w_{p+1} \dots w_{q-1}$ , and  $L_3$  for the segment  $w_q \dots w_{n+1}$ . By our assumptions, all three of these segments are decreasing sequences. In particular, since 2 comes before 3 in  $w$ , 2 cannot

appear in  $L_3$ . In fact, 2 and 3 cannot appear in the same segment. For convenience we are going to use bars between these segments.

First, we assume that  $p = 2$ . Since any element of  $X_{I,J}^+$  has descents (at least) at the positions  $J = \{1, \widehat{2}, 3, 4, \dots, \widehat{q}, \dots, n+1\}$ , the bottom element  $\tau_0$  has to be of the form

$$\tau_0 = 2 \ 1 \mid n+1 \ n \dots n-q+3 \ n-q+2 \ n-q+1 \dots 3, \quad (10)$$

or of the form

$$\tau_0 = n+1 \ n \dots n-q+4 \ 2 \ 1 \mid n-q+3 \ n-q+2 \dots 3. \quad (11)$$

Bars between numbers indicate the possible positions of ascents. Note that the number of inversions of the former permutation is  $1 + \binom{n-1}{2}$ , and the rank of the latter is

$$\begin{aligned} f_n(q) &:= \left( \sum_{i=1}^{q-2} n+1-i \right) + 1 + \left( \sum_{i=q+1}^n n+1-i \right) \\ &= \binom{n+1}{2} + 1 - (n+1-q) - (n+1-(q-1)). \end{aligned}$$

which is always greater than the former. Therefore, the minimal element  $\tau_0$  of  $X_{I,J}^+$  starts with  $2 \ 1$  (as in [10](#)).

This element has a single ascent at the 2nd position. We are going to analyze the covers of  $\tau_0$ . Since an upward covering in Bruhat order is obtained by moving a larger number to the front,  $n+1$  of  $L_2$  moves into  $L_1$  and accordingly either 2 or 1 from  $L_1$  moves into  $L_2$ .

Now, recall that each double coset  $W_I z W_J$  is an interval of  $W$  in Bruhat order and  $X_{I,J}^+$  consists of maximal elements of these intervals (see [\[2, Theorem 1.2\(ii\)\]](#)). It follows from this critical observation that, to obtain a covering of  $\tau_0$ , 1 has to move, and it becomes the last entry of  $L_2$ . In other words, permutation

$$\tau_1 = n+1 \ 2 \mid n \dots n-q+3 \ 1 \mid n-q+2 \ n-q+1 \dots 3$$

is the unique element in  $X_{I,J}^+$  that covers  $\tau_0$ .

Next, we analyze the covers of  $\tau_1$  which has only two possible coverings obtained as follows: 1) 2 moves into  $L_2$  and  $n$  moves into  $L_1$ , 2) 1 moves into  $L_3$  and  $n-q+2$  moves into  $L_2$ . The resulting elements are

$$\begin{aligned} \tau_2 &= n+1 \ n \mid n-1 \dots n-q+3 \ 2 \ 1 \mid n-q+2 \ n-q+1 \dots 3 \\ \tau_3 &= n+1 \ 2 \mid n \dots n-q+3 \ n-q+2 \mid n-q+1 \dots 3 \ 1 \end{aligned}$$

It is not difficult to see that each of these two elements are covered by the same element, namely

$$\tau_4 = n+1 \ n \mid n-1 \dots n-q+3 \ n-q+2 \ 2 \mid n-q+1 \dots 3 \ 1.$$

Observe that, in  $\tau_4$  the only element that can be moved is 2 and this is possible only if  $q \leq n-1$ . This agrees with our assumption on  $q$ . Therefore,  $\tau_4$  is covered by  $w_0$  only (in



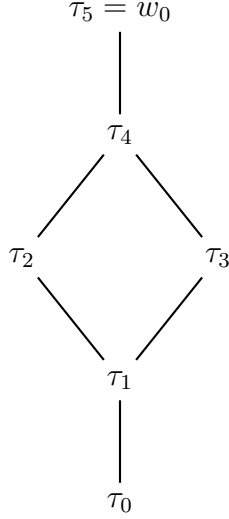


Figure 1: Bruhat order on  $X_{I,J}^+$  for type  $A_n$  (i)

$X_{I,J}^+$ ). Note that all that is said above is independent of  $n$  as long as  $p = 2$  and  $3 < q < n$ . Hence, our poset is as in Figure 1.

Finally for type  $A_n$  (i) with  $I^c = \{2\}$ , we look at the case for  $p > 2$ . The only difference between this and  $p = 2$  case is that the first  $p - 2$  terms of the elements of  $X_{I,J}^+$  all start with  $n + 1 \ n \ n - 2 \dots n - p$ . By induction, we reduce this case to  $p = 2$  and the poset  $X_{I,J}^+$  is in fact isomorphic to the one in Figure 1.

Now we assume that  $I^c = \{n - 1\}$  and  $J = \{p, q\}$  with  $2 \leq p < p + 1 < q \leq n - 1$ . As in the previous case, for an element  $w \in X_{I,J}^+$  these conditions imply that  $w$  is of the form  $w = L_1|L_2|L_3$ , where  $L_i$ ,  $i = 1, 2, 3$  are decreasing sequences of lengths  $p, q - p$  and  $n + 1 - q$ , respectively, and the number  $n - 1$  appears before  $n$  in  $w$ . It follows that the smallest element of  $X_{I,J}^+$  is of the form

$$\tau_0 = w_1 \dots w_p | w_{p+1} \dots w_q | w_{q+1} \dots w_{n+1} = n - 1 \ n - 2 \dots n - q \mid n + 1 \ n \ n - q - 1 \ n - q - 2 \dots 1$$

Then arguing exactly as in the previous case one sees that the poset under consideration is also of the form Figure 1.

## 4.2 Case 2.

We start with the case  $|I^c| = 1$ ,  $J^c = \{1, j\}$  with  $2 < j < n - 1$ . Here, we have more variety than the previous case. Let us first analyze the sub-case  $I^c = \{1\}$ . Once again we are going to use bars to indicate the positions of the ascents (defined by  $J^c$ ). As before, the strings between these bars are denoted by  $L_1, L_2$ , and  $L_3$ .

Suppose  $w = w_1 \dots w_{n+1}$  is the smallest element  $\tau_0$  of  $X_{I,J}^+$ . In  $w \in X_{I,J}^+$ , the number 1 comes before 2. If 1 is in  $L_2$ , then we can move it to  $L_1$  (and move the number in  $L_1$  to  $L_2$  at

an appropriate position) without altering the ascent positions, hence we stay in  $X_{I,J}^+$ . At the same time, such a move reduces the length in Bruhat order, contradicting the minimality of  $w$ . Therefore, 1 has to be in  $L_1$  to begin with. Since there are only two possible ascent positions one of which is at the 1st position and there is a 1 in it, by computing the length of the permutation  $w'$  that has a second ascent at the  $j$ th position we see that it cannot be smaller than

$$w = 1 \mid n+1 \ n \dots n-(j-1) \ n-j \dots 2, \quad (12)$$

in Bruhat order. Therefore,  $\tau_0$  is the element  $w$  in [12](#). In fact, there is a unique element in  $X_{I,J}^+$  that covers  $\tau_0$  whose unique ascent is at the  $j$ th position:

$$\tau_1 = n+1 \ n \dots n+2-j \ 1 \mid n-(j-1) \ n-j \dots 2.$$

Now, because no other ascent is allowed, there is only one way to move up in the Bruhat order by moving a smaller entry from the string  $L_3$  to  $L_2$  which in fact gives the longest permutation. Therefore, the poset of  $X_{I,J}^+$  is a chain of length 3:

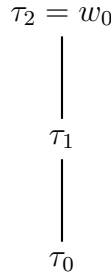


Figure 2: Bruhat order on  $X_{I,J}^+$  for type  $A_n$ ,  $I^c = \{1\}$ ,  $J^c = \{1, j\}$ .

Now we look at the general case  $I^c = \{i\}$  ( $1 < i \leq n$ ),  $J^c = \{1, j\}$  with  $2 < j < n-1$ . As the number  $i \in I^c$  grows (up to  $\lfloor \frac{n+1}{2} \rfloor$ ) we get more freedom to position  $i$  and  $i+1$ . This makes our posets grow taller. Rather than listing each case separately, which depend on the choice of  $i$  and  $j$ , we present the general combinatorial rule that governs the structure of  $X_{I,J}^+$ . The underlying idea of our description is already present in the previous cases.

Our first task is to decide for the smallest element  $\tau_0$  of  $X_{I,J}^+$ . Once again, a generic element  $w = w_1 \dots w_{n+1} \in X_{I,J}^+$  is viewed as a concatenation of three segments,  $w = L_1 L_2 L_3$  where  $L_1 = w_1$ ,  $L_2 = w_2 \dots w_j$ , and  $L_3 = w_{j+1} \dots w_{n+1}$ . Possible ascents are at the 1st and at the  $j$ th positions. At the same time, the number  $i$  appears before  $i+1$  in  $w$ , therefore,  $i$  and  $i+1$  are always contained in distinct segments (except in  $w_0$ ). In particular,  $i$  appears either in  $L_1$  or in  $L_2$ .

Assume that  $i \in L_2$  (hence  $i+1 \in L_3$ ) and let  $k$  denote the number in  $L_1$ . In particular, we know that  $k \notin \{i, i+1\}$ . By interchanging  $k$  by  $k-1$  we move down in the Bruhat order and still stay in  $X_{I,J}^+$ , which contradicts the minimality of  $\tau_0$ . Therefore, the only possibility for  $k$  is  $k = 1$ . But this puts us back in the case  $I^c = \{1\}$ ,  $J^c = \{1, j\}$ , and there is not

much ado. Therefore, we assume that  $i \in L_1$ . Now there are two cases each which is easy to verify:

1)  $j \leq i$  and  $\tau_0$  is of the form

$$\tau_0 = i \ i - 1 \dots i - j + 1 \mid n + 1 \ n \dots i + 1 \ i - j \ i - j - 1 \dots 1. \quad (13)$$

2)  $j > i$  and  $\tau_0$  is of the form

$$\tau_0 = i \ n + 1 \ n \dots n + 2 - (j - i) \ i - 1 \ i - 2 \dots 1 \mid n + 1 - (j - i) \ n - (j - i) \dots i + 1 \quad (14)$$

Now we make some obvious observations regarding how the posets climbs up in Bruhat order on  $X_{I,J}^+$  starting with  $\tau_0$ 's as in 1) and 2). First of all, if  $\tau_0$  is as in 1), then to get a covering relation there is only one possible interchange, namely, moving  $i - j + 1 \in L_2$  into  $L_3$ . In this case, to maintain the descents, the number that is replaced by  $i - j + 1$  has to be  $n + 1$ , which goes into the first entry of  $L_2$ . In other words, the unique  $w \in X_{I,J}^+$  that covers  $\tau_0$  is

$$w = i \ n + 1 \ i - 1 \dots i - j + 2 \mid n \dots i + 1 \ i - j + 1 \ i - j \ i - j - 1 \dots 1. \quad (15)$$

It is easy to verify that there are exactly two elements that covers  $w$ :

$$w_{(2)} = n + 1 \ i \ i - 1 \dots i - j + 2 \mid n \dots i + 1 \ i - j + 1 \ i - j \ i - j - 1 \dots 1. \quad (16)$$

$$w^{(2)} = i \ n + 1 \ n \ i - 1 \dots i - j + 3 \mid n - 1 \dots i + 1 \ i - j + 2 \ i - j + 1 \dots 1. \quad (17)$$

**Remark 4.1.** Suppose  $w \in S_n$  is a permutation on  $[n] := \{1, \dots, n\}$  whose one-line notation ends with the decreasing string  $k \ k - 1 \dots 2 \ 1$ . Then any element in the upper interval  $[w, w_0] \subset S_n$  has the same ending. In other words, if  $w' \in [w, w_0]$ , then the last  $k$  entries of  $w'$  are exactly  $k, k - 1, \dots, 1$  in this order. Similarly, if  $w$  begins with the decreasing string  $n \ n - 1 \dots k$  for some  $k \in [n]$ , then any element in the upper interval  $[w, w_0] \subset S_n$  has the same beginning. So, essentially these elements form an upper interval in  $S_{n-k}$ .

By Remark 4.1 we see that the elements that are above  $w_{(2)}$  in  $X_{I,J}^+$  all start with  $n + 1$ . Also, since there is no ascent at the 1st position for such elements, we see that the resulting upper interval  $[w_{(2)}, w_0]$  in  $X_{I,J}^+$  is isomorphic to a similar double coset poset in  $S_n$  with  $I^c = \{i\}$  and  $J^c = \{j\}$ , hence it is a chain.

There are two covers of  $w^{(2)}$ ; one of them,  $w_{(3)}$ , is an element of the interval  $[w', w_0]$  (hence  $w_{(3)}$  covers  $w'$  as well). The other cover of  $w^{(2)}$  is

$$w^{(3)} = i \ n + 1 \ n \ n - 1 \ i - 1 \dots i - j + 4 \mid n - 1 \dots i + 1 \ i - j + 3 \ i - j + 2 \dots 1. \quad (18)$$

Now the pattern is clear;  $w^{(3)}$  has exactly two covers one of which lies in  $[w_{(3)}, w_0]$  and the other  $w^{(4)}$  has a similar structure to  $w^{(3)}$ . Therefore, the resulting poset is a ladder, as

depicted in Figure 3, and the chains  $w^{(p)}$  and  $w_{(p)}$ ,  $p \geq 3$  will climb up to meet first time either at  $w_0$  or at

$$w^{(m+1)} = w_{(m+1)} = n + 1 \mid n \dots n + 1 - (j - 3) \mid i \mid n + 1 - (j - 4) \dots \hat{i} \dots 2 \mid 1 \quad (19)$$

In the latter case, of course,  $w_0$  is the unique cover of  $w^{(m+1)} = w_{(m+1)}$ . In particular, the height of our poset does not exceed  $j$ .

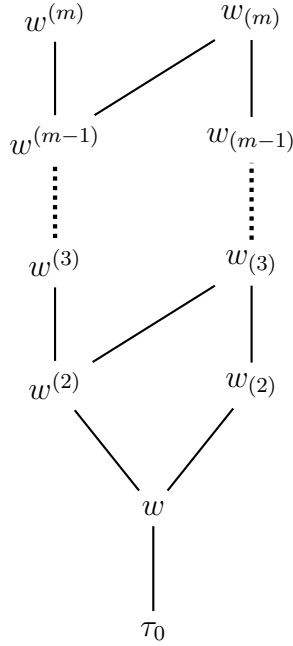


Figure 3: Bruhat order on  $X_{I,J}^+$  for type  $A_n$ ,  $I^c = \{2\}$ ,  $J^c = \{1, j\}$

The remaining case is  $I^c = \{i\}$  and  $J^c = \{j, n\}$  with  $2 < j < n - 1$ . The proofs and the results are identical to the previous case. The poset is either a chain when  $I^c = \{1\}$  and  $j$  arbitrary, or it is isomorphic to one of the ladder posets (depicted below).

In summary, we proved our main result.

**Theorem 4.2.** The poset of  $G$  orbit closures in the spherical product  $G/P_I \times G/P_J$  is either a chain or one of the “ladder lattices” as depicted in Figure 4.

We anticipate that the following is true for other types: if  $|J^c| = 1$ , then the poset of  $G$  orbit closures is a chain, otherwise it is one of the ladder lattices as in Figure 4. These cases will be handled in an upcoming manuscript.

## 5 Final remarks

The  $P_I$  orbits in  $G/P_J$  are parametrized by  $W_I$  orbits in  $W/W_J$ . Now that we understand how these orbits fit together in Bruhat order, next we are going to describe the Bruhat order

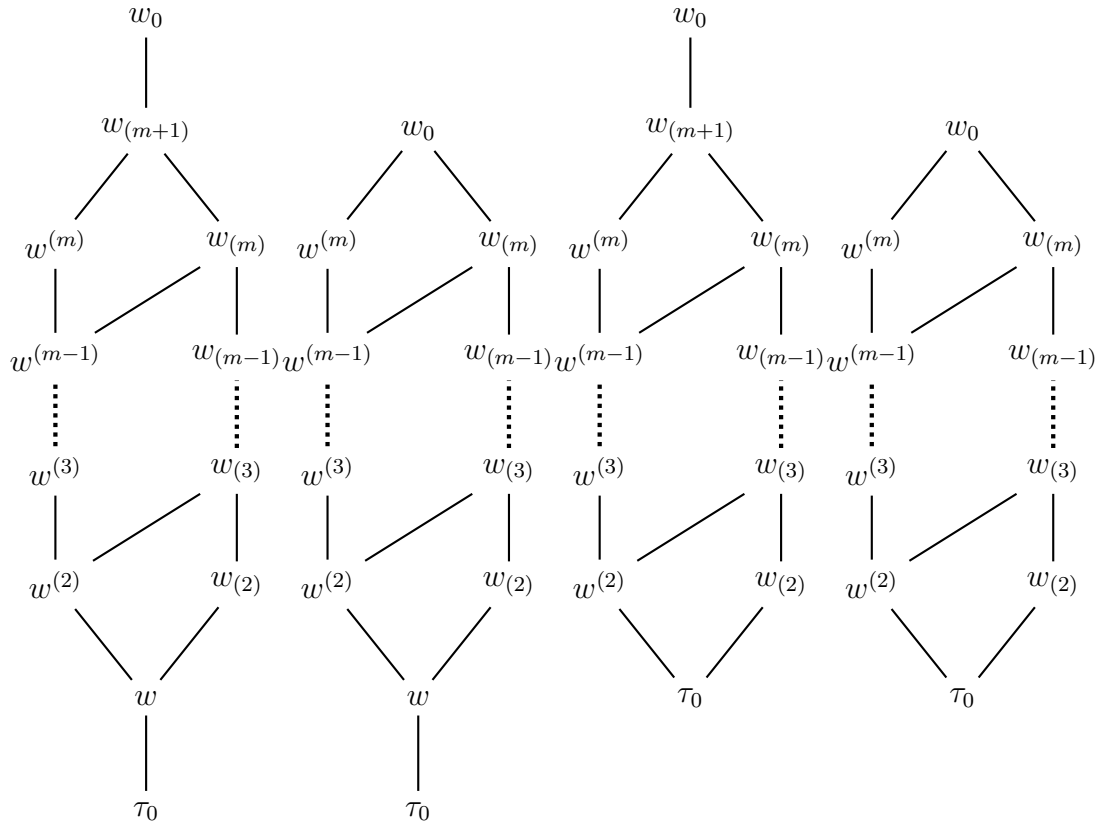


Figure 4: Possible ladder posets describing Bruhat orders on  $X_{I,J}^+$  for type  $A_n$

on  $B$  orbits in  $X = P_I \backslash G \times G / P_J$ . By using the  $G$  equivariant isomorphism defined in 2 and the Bruhat decomposition  $P_I = BW_I B$ , we see that  $B$  orbits in  $X$  are parametrized by the union of the elements of  $W_I$  orbits in  $W/W_J$ .

We already know that the set  $X_{I,J}^-$  is a parametrization of  $(W_I, W_J)$  double coset representatives in  $W$ . In particular, if  $w \in X_{I,J}^-$ , then  $w \in W^J$ . Recall also that, if we define  $H := I \cap wJw^{-1}$ , then each element  $x \in W_I w W_J$  has a unique expression of the form  $x = uwv$ , where  $u \in W_I$  is a minimal length coset representative for  $W_I/W_H$ ,  $v \in W_J$  and the following equality is satisfied:  $\ell(x) = \ell(u) + \ell(w) + \ell(v)$ . In particular, we see that the left  $W_I$  orbits in  $W_I w W_J$  are parametrized by the minimal length coset representatives  $W_I/W_H$ .

From now on, we denote the set of such  $u \in W$  by  $W^{I,J}$ . Note that  $W^{I,J}$  is in bijection with the cosets  $W_I w / W_J$ . The proof of our next result is an easy consequence of these observations.

**Theorem 5.1.** The set of all  $B$  orbits in  $P_I \backslash G \times G / P_J$  is parametrized by the set

$$D_{I,J}^- := \{(u, w) : u \in W^{I,J}, w \in X_{I,J}^-\}.$$

Let  $x = (u_1, w_1)$  and  $y = (u_2, w_2)$  be two elements from  $D_{I,J}^-$  and let  $O_x, O_y$  denote the corresponding  $B$  orbit closures in  $X$ . Then

$$\begin{aligned} O_x \subset O_y &\iff u_1 w_1 \leq u_2 w_2 \\ &\iff u_1 \leq u_2 \text{ and } w_1 \leq w_2. \end{aligned}$$

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